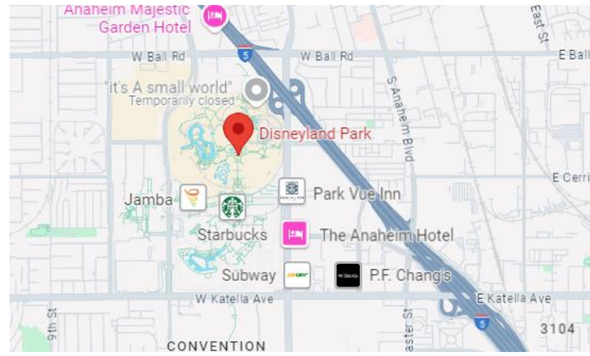
CH NN –DISTANCE ON THE LINE AND IN THE PLANE



One important factor when visiting Disneyland is the **distance** between your home and the theme park. It may determine your mode of transportation, the travel time, and thus the total cost of the trip.

You might want to review the chapter *Absolute Value, the Basics* to help you understand this chapter.

❑ *DISTANCE ON A LINE*

> *Distance* is never negative.

Our plan now is to create a formula that will give us the ***distance*** between two points on a line. Consider the two points 10 and 17 on a line. Is it pretty clear that the distance between them is 7? If it's not really obvious, you can simply subtract the smaller number from the larger one:

$$\boldsymbol{larger - smaller} = 17 - 10 = \mathbf{7} \checkmark$$

This formula (*larger – smaller*) works perfectly for any two numbers on a line:

The distance between –7 and 5 = 5 – (–7) = 5 + 7 = **12**.
[Note that 5 is larger than –7.]

The distance between –9 and –20 = –9 – (–20) = –9 + 20 = **11**.

[Note that –9 is larger than –20.]

The distance between 12 and 12 = 12 – 12 = **0**.

[Note that it doesn't matter which you choose as the larger number.]

NOTE: Subtracting in the wrong order (*smaller – larger*) is catastrophic. For example, if we try to find the distance between 10 and 17 like this:

10 – 17 = –7

we get a negative distance — quite forbidden!

Our way out of this mess is to come up with a *formula* for the distance between any two points on a line. In other words, we need a formula for the distance between the numbers ***a*** and ***b*** on a line, when we may not know which one of them, *a* or *b*, is the larger one. [See the NOTE above.]

Here's the secret: Use ***absolute value***. Then, if we were to "accidentally" subtract in the wrong direction — and end up with a negative distance (which DOESN'T EXIST) — the absolute value will automatically convert that negative number into a positive number.

Moreover, if we properly subtract *larger – smaller*, we'll get a positive number, whose absolute value will be just that number. And if the two numbers are the same, the absolute value of 0 is still 0. No matter the sizes of the two numbers, subtracting them (regardless of which one's bigger, or even if they're the same) — and then applying the absolute value to the difference — works every time.

So, in short, if ***a*** and ***b*** are any two numbers on the number line, we don't even have to worry about which one is bigger. We calculate the distance between them by using the following formula:

$$d = |a - b|$$

The distance between two points on the line is the absolute value of their difference.

Four Examples:

A. The distance between 7 and 3 is $|7-3| = |4| = \mathbf{4}$

B. The distance between 10 and 25 is $|10-25| = |-15| = \mathbf{15}$

C. The distance between π and π is $|\pi-\pi| = |0| = \mathbf{0}$

D. The distance between –17 and –5 is

$$|-17-(-5)| = |-17+5| = |-12| = \mathbf{12}$$

Homework

1. Find the **distance** between the given pair of points on the number line, using the absolute-value formula, and show each step:

a. 7 and 2 b. –2 and 9 c. –3 and –3

d. 99 and –99 e. –5 and –13 f. –20 and –4

❑ *DISTANCE IN THE PLANE [USING A TRIANGLE]*

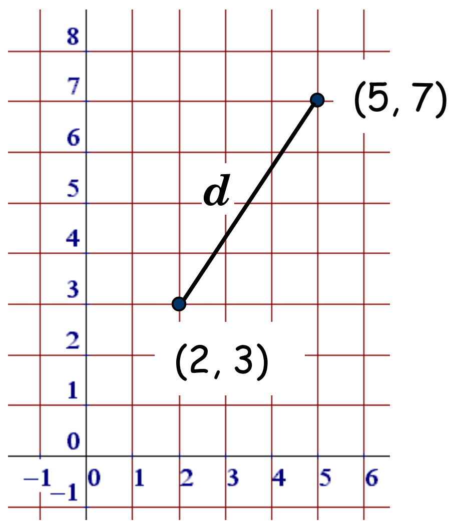
Assuming that you know how to plot points in the plane and remember the Pythagorean Theorem, we can tackle the question:

How do we find the ***distance*** *between two points in the* ***plane****?*

If the Earth were flat, it would be like asking how far apart two cities are if we know the latitude and longitude of each city.

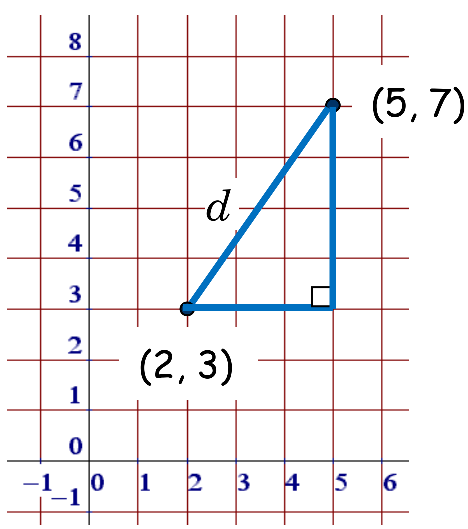
<u>EXAMPLE 1:</u> **Find the distance between the points (2, 3) and (5, 7) in the plane.**

<u>Solution:</u> Let's draw a picture and see what we can see. We'll plot the two given points and connect them with a straight line segment. The distance between the two points, which we'll call $\boldsymbol{d}$, is simply the length of that line segment.



How far is it between the two points (2, 3) and (5, 7)? Equivalently, what is the length (d) of the line segment connecting the two points?

Now what do we do? Well, here comes the interesting part. If we're creative enough, we might see that the segment connecting the two points can be thought of as the *hypotenuse* of a right triangle — as long as we sketch in a pair of legs to create such a triangle. Let's do that:



We've created a right triangle whose legs have lengths 3 and 4, and whose hypotenuse has a length equal to the distance between the two given points.

Sure enough, we've

constructed a right triangle where d is the length of the hypotenuse. If we can determine the lengths of the legs, then we can use the Pythagorean Theorem to find the length of the hypotenuse. By counting squares along the base of the triangle, we see that one leg is 3. Similarly, the other leg (the height) is 4. Since the square of the hypotenuse is equal to the sum of the squares of the legs, we can write the equation

$$
\begin{aligned}
& d^2 = 3^2 + 4^2 && \text{(Pythagorean Theorem: } \text{hyp}^2 = \text{leg}^2 + \text{leg}^2\text{)} \\
\Rightarrow\ & d^2 = 9 + 16 && \text{(square the legs)} \\
\Rightarrow\ & d^2 = 25 && \text{(add)} \\
\Rightarrow\ & d = 5 && \text{(since } \sqrt{25} = 5\text{)}
\end{aligned}
$$

You may notice that $d = -5$ also satisfies the equation $d^2 = 25$, since $(-5)^2 = 25$; but does a negative value of d make sense? No, because distance can never be negative; so we conclude that

The distance between the two points is **5**

Homework

2. By plotting the two given points in the plane and using the Pythagorean Theorem, find the **distance** between the points:

   a. (1, 1) and (4, 5)

   b. (2, –3) and (6, –6)

   c. (–3, 5) and (2, –7)

   d. (–4, –5) and (1, 7)

   e. (–5, 0) and (1, 8)

   f. the origin and (6, 8)

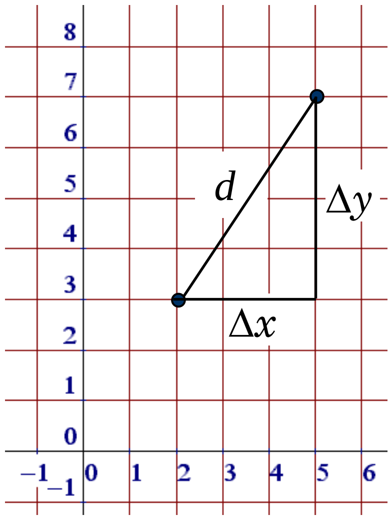
   g. the origin and (–5, –12)

   h. (2, 5) and (2, –1)

   i. (–3, 4) and (2, 4)

   j. (π, 99) and (π, 99)

❑ DISTANCE IN THE PLANE [USING A FORMULA]



To find the distance between two points in the plane, we've learned to create a right triangle so that the hypotenuse is the distance between the points.

Notice that the bottom leg is the change in x (Δx), while the vertical leg is the change in y (Δy). Thus, by employing the Pythagorean Theorem, we know that $d^2 = (\Delta x)^2 + (\Delta y)^2$. Solving this equation for d (and ignoring the negative square root), results in our Distance Formula:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

<u>EXAMPLE 2:</u> **Find the distance between the points (2, –8) and (12, –14) in the plane.**

<u>Solution:</u> Using the formula we just created (and with no reference to a graph), we can calculate the following:

$$\boldsymbol{\Delta x} = 2 - 12 = \mathbf{-10}$$

$$\boldsymbol{\Delta y} = -8 - (-14) = -8 + 14 = \mathbf{6}$$

So the distance between the points is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

<u>Note:</u> If we calculate Δx or Δy by subtracting in the reverse order, it doesn't matter in the distance formula, since these changes are being squared anyway.

$$\Rightarrow \quad d = \sqrt{(-10)^2 + (6)^2}$$

$$\Rightarrow \quad d = \sqrt{100 + 36}$$

$$\Rightarrow \quad d = \sqrt{136}$$

$$\Rightarrow \quad d = \boxed{2\sqrt{34}}$$

❑ *THE CLASSIC DISTANCE FORMULA*

An alternative formula, commonly found in books, is to realize that Δx is the difference of the x-values, and Δy is the difference of the y-values, so that the distance between the points (x_1, y_1) and (x_2, y_2) is

$$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Homework

3. Use either the Distance Formula with the Δx and Δy in it or the classic Distance Formula to find the **distance** between the given pairs of points:

a. (1, 1) and (4, 6)

b. (–2, –3) and (6, –6)

c. (–3, 5) and (0, –7)

d. (–4, 5) and (1, 7)

e. (–9, 0) and (1, 8)

f. the origin and (6, 11)

g. the origin and (–4, –10)

h. (2, 5) and (2, –1)

i. (–3, 4) and (2, 4)

j. $(\pi, \sqrt{2})$ and $(\pi, \sqrt{2})$

❑ To ∞ and Beyond

Find the **distance** in 3-space between the points (1, –5, 8) and (–4, 9, 9).

Solutions

1. a. $|7-2| = |5| = 5$ b. $|-2-9| = |-11| = 11$

c. $|-3-(-3)| = |-3+3| = |0| = 0$

d. $|99-(-99)| = |99+99| = |198| = 198$

e. 8 f. 16

2. a. 5 b. 5 c. 13 d. 13 e. 10

f. 10 g. 13 h. 6 i. 5 j. 0

3. a. $\sqrt{34}$ b. $\sqrt{73}$ c. $3\sqrt{17}$ d. $\sqrt{29}$ e. $2\sqrt{41}$

f. $\sqrt{157}$ g. $2\sqrt{29}$ h. 6 i. 5 j. 0

"<u>Give</u> a man a fish
and you feed him for a day.
<u>Teach</u> a man to fish
and you feed him for a lifetime."

Chinese Proverb